Range CUBE: Efficient Cube Computation by Exploiting Data Correlation

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Abstract

Data cube computation and representation are prohibitively expensive in terms of time and space. Prior work has focused on either reducing the computation time or condensing the representation of a data cube. In this paper, we introduce Range Cubing as an efficient way to compute and compress the data cube without any loss of precision. A new data structure, range trie, is used to compress and identify correlation in attribute values, and compress the input dataset to effectively reduce the computational cost. The range cubing algorithm generates a compressed cube, called range cube, which partitions all cells into disjoint ranges. Each range represents a subset of cells with the identical aggregation value as a tuple which has the same number of dimensions as the input data tuples. The range cube preserves the roll-up/drill-down semantics of a data cube. Compared to H-Cubing, experiments on real dataset show a running time of less than one thirtieth, still generating a range cube of less than one ninth of the space of the full cube, when both algorithms run in their preferred dimension orders. On synthetic data, range cubing demonstrates much better scalability, as well as higher adaptiveness to both data sparsity and skew.

1 Introduction

Data warehousing is an essential element for decision support technology. Summarized and consolidated data is more important than detailed and individual records for knowledge workers to make better and faster decisions. Data warehouses tend to be orders of magnitude larger than operational database in size since they contain consolidated and historical data. Most workloads are complex queries which access millions of records to perform several scans, joins and aggregates. The query response times are more critical for online data analysis applications. A warehouse server typically presents multi-dimensional views of data to a variety of analysis applications and data mining applications to facilitate complex analyses. These applications require grouping by different sets of attributes. The data Cube was proposed[7] to precompute the aggregation for all possible combination of dimensions to answer analytical queries efficiently. For example, consider a sales data warehouse: `(Store, Date, City, Product, Price)`. Attributes `Store, Date, City` and `Product` are called *dimensions*, which can be used as grouped by attributes. `Price` is a numeric *measure*. The dimensions together uniquely determine the measure, which can be viewed as a value in the multidimensional space of dimensions. A data Cube provides aggregation of measures for all possible combinations of dimensions. A *cuboid* is a group-by of a subset of dimensions by aggregating all tuples on these dimensions. Each cuboid comprises a set of *cells*, which summarize over tuples with specific values on the group-by dimensions. A cell $a = (a_1, a_2, ..., a_n, measures_a)$ is an $m - dimensional$ cell if exactly m values among $a_1, a_2, ..., a_n$ are not $*$. Here is an example:

Example 1 *Consider the base table given below.*

Store	*City*	*Product*	*Date*	*Price*
Gateway	*NY*	*DESKTOP*	*12/01/2000*	*$ 100*
CompUSA	*NY*	*DESKTOP*	*12/02/2000*	*$ 100*
CompUSA	*SF*	*LAPTOP*	*12/03/2000*	*$ 150*
Staple	*LA*	*LAPTOP*	*12/03/2000*	*$ 60*

The cuboid `(Store, *, *, *)` contains three $1 - dimensional$ cells: `(Gateway, *, *, *)`, `(CompUSA, *, *, *)`, and `(Staple, *, *, *)`. `(CompUSA, NY, *, *)` and `(CompUSA, SF, *, *)` are $2 - dimensional$ cells in cuboid `(Store, City, *, *)`.

The data cube computation can be formulated as a process that takes a set of tuples as input, computes with or without some auxiliary data structure, and materializes the aggregated results for all cells in all cuboids. A native representation of the cube is to enumerate all cells. Its size is usually much larger than that of input dataset, since a table with n dimension results in 2^n cuboids. Thus, most work is dedicated to reduce either the computation time or the final cube size, such as efficient cube computation[9, 4, 17, 12], cube compression[13, 10, 15]. These cost reduction are all without loss of any information, while some others like approximation[3, 2], iceberg cube[6] reduce the costs by skipping "trivial" information. The approaches without losing information can be classified into three categories (Figure 1) in terms of how the data is compressed before and after the computation.

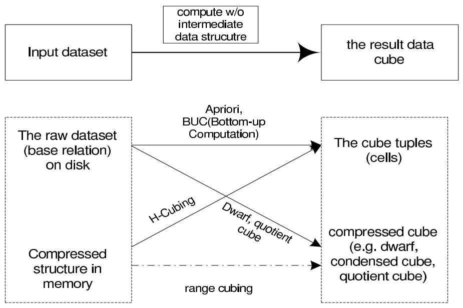

compute w/o intermediate data strucutre
Input dataset
the result data cube
Apriori, BUC(Bottom-up Computation)
The raw dataset (base relation) on disk
The cube tuples (cells)
H-Cubing
Dwarf, quotient cube
Compressed structure in memory
compressed cube (e.g. dwarf, condensed cube, quotient cube)
range cubing

Figure 1. Classification of recent works on cube computation

The Apriori-like algorithm[1] and the Bottom-UP Computation (BUC)[4] approaches are the most influential works for both iceberg cube and full cube computation. They compute a data cube by scanning a dataset on disk and generate all cells one by one on disk. BUC is especially targeted for sparse datasets, which is usually the case with real data. The performance deteriorates when applied to skewed or dense data.

Han et al. [9] devise a new way for cube computation by introducing a compressed structure of the input dataset, H-Tree. The dataset is compressed by prefix sharing to be kept in memory. It also generates cells one by one and does well on dense and skewed dataset. It can be used to compute both iceberg cubes and full cubes, and is shown to outperform BUC when dealing with dense or skew data in iceberg cube computation. Xin et al. [16] will publish a new method called star-cubing to traverse and compute on the star-tree, which is basically an H-Tree without side links, to further improve the computational efficiency.

Another set of work mainly aims to reduce the cube size. It also scans the input dataset, but generates a compressed cube. Sismanis et al. [14] use the CUBEtree or compressed CUBEtree to compress the full cube in a tree-like data structure in memory, by utilizing prefix sharing and suffix coalescing. It requires the memory to accomodate the compressed cube. Some work[15][10] condenses the cube size by coalescing cells with identical aggregation values. Wang et al. [15] use "single base tuple" compression to generate a "condensed cube". Lakshmanan et al. [10] propose a sematics-preserving compressed cube to optimally collect cells with the same aggregation value. The compressed cube computation is based on BUC algorithm, whose time efficiency is comparable to BUC[4]. Furthermore, they index the "classes" of cells using a QC-tree[11].

However, these works are all not designed to reduce both computational time and output I/O time to achieve the reduction on overall speed-up. As observed in most efficient cube computation works[16, 4, 12], the output I/O time dominates the cost of computation in high dimension and high cardinality datasets since the output of a result cube is extremely large, while most compression algorithms focus on how to reduce the final cube size.

In this paper, we propose an efficient cube computation method to generate a compressed data cube, which is both sematics-preserving and format-preserving. It cuts down both computational time and output I/O time without loss of precision. The efficiency of cube computation comes from the three aspects:(1)It compresses the base table into a range trie, a compressed trie, so that it will calculate cells with identical aggregation values only once. (2)The travesal takes advantage of simultanous aggregation, that is, the $m - dimensional$ cell will be computed from a bunch of $(m + 1) - dimensional$ cells after the initial range trie is built. At the same time, it facilitates Apriori pruning. (3) The reduced cube size requires less output I/O time. Thus, we reduce the time cost not only by utilizing a data structure to efficiently compute the aggregation values; but also by reducing the disk I/O time to output the cube.

The motivation of the approach comes from the obervation that real world datasets tend to be correlated, that is, dimension values are usually dependent on each other. For example, `Store` "Starbucks" always makes `Product` "Coffee". In the weather dataset used in our experiments, the `Station Id` will always determine the value of `Longitude` and `Latitude`. The effect of correlated data on the corresponding data cube is to generate a large number of cells with the same aggregation values. The basis of our approach is to capture the correlation or dependency among dimension values by using a range trie, and generating a range representing a sequence of cells each time.

1.1 Related Work

We will give an overview of those works most related to our approach.

1. H-Cubing and Star-Cubing

H-Cubing and Star-Cubing both organize input tuples in a hyper-tree structure. Each level represents a dimension, and the dimension values of each base tuple are populated on the path of depth d from the root to a leaf. In an H-Tree, nodes with the same value on the same level will be linked together with a side link. A head table is associated with each H-Tree to keep track of each distinct value in all dimensions and link to the first node with that value in H-Tree. The data structure is revised in an incoming paper[16] as a star tree, which excludes the side links and head tables in an H-Tree. Also a new traversal algorithm is proposed to take advantage of both simultanous aggregation and pruning efficiency. Moreover, it prunes the base table using a star table before it is loaded into the start tree to accelerate the computation of iceberg cubes. The efficiency of the new star cubing algorithm lies in its way to compute the cells in some order of cuboids, so that it can enjoy both the sharing of computation as Array Cube[17] and similar pruning efficiency as BUC.

2. Condensed Cube and Quotient Cube

Condensed cube[15] is a method to condense the cube using the notion "single tuple cell", which means that a group of cells will contain only one tuple and share the same aggregation value as the tuple's measure value. Quotient cube generalizes the idea to group cells into classes. Each class contains all cells with the identical aggregation values. Both of them extend the BUC algorithm to find "single tuple cells" or "classes" during the bottom-up computation of a full cube.

Our approach integrates the benefits of these two categories of algorithms to achieve both computational efficiency and space compression. It compresses base table without loss of information into a compressed trie data structure,called range trie, and computes and generates the subsets of cells with identical aggregation values together as a range. The cube computation algorithm enjoys both sharing of computation and pruning efficiency for iceberg cube.

The rest of paper will be organized as follows: Section 2 provides some theoretical basis of our work. Section 3 describes the *range trie* data structure and construction procedure, followed by section 4 showing how the range cube is represented. Section 5 presents the range cubing algorithm. Section 6 analyzes the experimental results to illustrate the benefits of the proposed method. We conclude the work in Section 7.

2 Background

We provide some notions needed later in this section. A *cuboid* is a multi-dimensional summarization of a subset of dimensions and contains a set of cells. A data cube can be viewed as a lattice of cuboids, which contains a set of cells.

Now, we define the "$\preceq$" relation on the set of cells of a data cube.

Definition 1 *If $a = (a_1, a_2, ..., a_n)$ is $ak - dimensional$ cell and $b = (b_1, b_2, ..., b_n)$ is an $l - dimensional$ cell in an n-dimensional cube, define the relation "$\preceq$" between a and b:*

$$a \preceq b \textit{ iff } k \geq l \textit{ and } a_i = b_i \; \forall \; b_i \neq * \textit{ and } 1 \leq i \leq n.$$

If S_{cells} is the set of cells in a cube, $(S_{cells}, \preceq)$ is a partially ordered set as shown in the next lemma.

Lemma 1 *The relation "$\preceq$" on the set of cells in a cube is a* ***partial order****, and the set of cells together with the relation is a* ***partially ordered set****.*[1]

The proof is to show the relation "$\preceq$" on the set of cells is reflexive, antisymmetric, and transitive. It is straightforward.

Intuitively, cells $a \preceq b$ means that a can roll-up to b and the set of tuples to compute the cell a is a subset of that to compute the cell b.

Example 2 *Consider the base table in Figure 2(a). Figure 2(b) is the lattice of cuboids in the data CUBE of our example.*

From the definition of the containment relation "$\preceq$", `(S1,C1,*, *)` $\preceq$ `(S1,*,*,*)`. Also `(S1,*,P1,*)` $\preceq$ `(S1,C1,P1,*)` $\preceq$ `(S1,C1,P1,D1)`.

Then we introduce the notion of *range* which is based on the relation "$\preceq$".

Definition 2 *Given two cells c_1 and c_2, $[c_1, c_2]$ is a* ***range*** *if $c_1 \preceq c_2$. It represents all cells, c, such that $c_1 \preceq c \preceq c_2$. we denote it as $c \in [c_1, c_2]$.*

Then in the above example, the range [`(S1,*,P1,*)`, `(S1,C1, P1,D1)`] contains four cells: `(S1,*,P1,*)`, `(S1,C1,P1,*)`, `(S1,*, P1,D1)`, and `(S1,C1,P1,D1)`. Moreover, all four cells in the range have identical aggregation value. So it means that we can use a range to represent these four cells without loss of information.

Lakshmanan et al. [10] developed a theorem to prove the sematic-preserving property of a partition of data cube based on the notion *convext partition*.

Definition 3 *The* ***Convex[5] Partition*** *of a partial ordered set is a partition which contains the whole range $[a, b]$ if and only if it contains a and b, and $a \preceq b$.*

If we define a partition based on "range", then it must be convex. As we will show later, a range cube is such a convex partition consisting of a collection of "ranges".

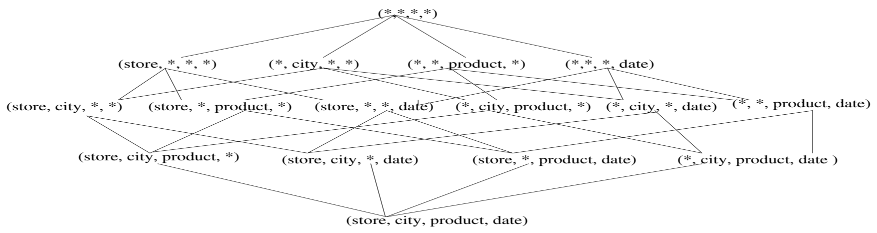
3 Range Trie

A range trie is a compressed trie structure constructed from the base table. Each leaf node represents a distinct tuple like star tree or H-Tree. The difference is that a range

[1]some papers[10] regard it a lattice of cells, but it is not unless we add a virtual empty node $\preceq$ all base nodes

Store	City	Product	Date	Price
S1	C1	P1	D1	$ 100
S1	C1	P2	D2	$ 50
S2	C2	P1	D2	$ 200
S2	C1	P1	D2	$ 120
S2	C3	P2	D2	$ 40
S3	C3	P3	D1	$ 250

(a) the base table



(b) the lattice of cuboids derived from the base table in (a)

Figure 2. The example for cells and cuboids for a base table

trie allows several shared dimension values as the key of each node, instead of one dimension value per node. The subset of dimension values stored in a node is shared by all tuples corresponding to the leaf nodes below the node. Therefore, each node represents a distinct group of tuples to be aggregated.

Two types of information are stored in a range trie: dimension values and measure values. The dimension values of tuples determine the structure of a range trie, and are stored in the nodes along paths from the root to leaves. The measure values of tuples determine the aggregation values of the nodes. A traversal from the root to a leaf node reveals all the information for any tuple(s) with a distinct set of dimension values represented by this leaf node. Each node contains a subset of dimension values as its **key**, which is shared by all tuples in its sub-trie. Consequently, its aggregation value is that of these tuples.

Consider a range trie on an n-dimensional dataset, ordered as $A_1, A_2, ..., A_n$. The **start dimension of the range trie** is defined as the smallest of these n dimensions, which is A_1. The smallest of all dimensions appearing in node i and its descendants is chosen as **the start dimension of node** i. So, if the key of a node i in the range trie is the values on a subset of the dimensions, (A_{i1}, A_{i2}, ..., A_{ik}), where $i1 < i2 < ... < ik$, the start dimension of node i is the smallest one A_{i1}. In the example shown in Figure 3(c), the start dimension of the range trie is `Store` and the start dimension of the node `(S1, C1)` is `Store`.

Each node can be regarded as the root of a sub-trie defined on a sequence of dimensions appearing in all its descendants. That is, node `(S1, C1)` is the root of a sub-trie defined on dimensions `(Product, Date)` of two tuples. We have the following relationship between the start dimension of a range trie and that of a node. We assume the key of the root of a range trie is empty. If it is not empty, we exclude those dimensions appearing in the root from the dimensions of the range trie. Since the dimension values in the root are common to all tuples, they are trivial to be considered.

Proposition 1 *The child nodes with the same parent node have the same start dimension. The start dimension of a range trie is the start dimension of its first level child node.*

Figure 3(c) shows a range trie constructed from the table in Example 2. The dimension order is `Store, City, Product, Date`. Two data tuples are stored below the node `(S1, C1,$150): (P1, D1,$100)` and `(P2, D2, $50)`. For simplicity, we omit the aggregation values for each node. The number in each node is the number of tuples stored below the node. The start dimension of the node `(P1, D1)` is `Product`, which is the smallest dimension among all the dimensions left. These two child nodes of the sub-trie rooted at `(S1, C1)` has the same start dimension `Product`, but distinct values `P1` and `P2`.

We now present a formal definition of a range trie.

Definition 4 *A **range trie** on dimensions $A_{i1}, A_{i2}, ..., A_{in}$ is a tree constructed from a set of data tuples. If the set is empty, the range trie is empty. If it contains only one tuple, it is a tree with only one leaf, whose key is all its dimension values. If it has more than one tuple, it satisfies the following properties:*

1. *Inheritance of ancestor Keys: If the key of the root node is $(a_{j1}, ..., a_{jr})$, all the data tuples stored in the range trie have the same dimension values $(a_{j1}, ..., a_{jr})$.*

2. *Uniqueness of Siblings' start dimension values: Children have distinct values on their start dimension.*

3. *Recursiveness of the definition: Each child of the root node is the root of a range trie on the subset of dimensions $\{A_{i1}, ..., A_{in}\}$ - $\{A_{j1}, ..., A_{jr}\}$.*

We can infer the following properties of a range trie from its definition:

1. The maximum depth of the range trie is the number of dimensions *n*.

2. The number of leaf nodes in a range trie is the number of tuples with distinct dimension values, which is bounded by the total number of tuples.

3. Because siblings have distinct values on their start dimension, the fan-out of a parent node is bounded by the cardinality of the start dimension of its child nodes.

4. Each interior node has at least two child nodes, since its parent node contains all dimension values common to all child nodes.

5. Each node represents a set of tuples represented by the leaf nodes below it, and we will show later that the set of tuples is used to compute a sequence of cells.

A range trie captures data correlation by finding dimension values that imply other dimension values. We now formalize the notion of data correlation in the following.

Definition 5 *We take the input dataset as the scope. When any tuples in the scope with the values $a_k, ..., a_l$ on the dimensions $A_k, ..., A_l$ always have the values $a_s, ..., a_t$ on the dimensions $A_s, ..., A_t$, it means that $a_k, ..., a_l$ imply $a_s, ..., a_t$, denoted as $(a_k, ..., a_l) \rightarrow \{a_s, ..., a_t\}$.*

It is clear to see that if $a_k, ..., a_l$ imply $a_s, ..., a_t$, then any superset of $a_k, ..., a_l$ will imply $a_s, ..., a_t$. According to this notation, we have `(S1)` $\rightarrow$ {`C1`} and `(S1, P1)` $\rightarrow$ {`C1, D1`} in the example.

Lemma 2 *Given a node with the key $(a_{i1}, a_{i2}, ..., a_{ir})$, where a_{i1} is the start dimension value of the node, the start dimension values of the node and its ancestors imply the dimension values $a_{i2}, ...$ and a_{ir}.*

The intuition is that the dimension values in a node are implied by the dimension values of its ancestors and its start dimension value. The dimension values of its ancestors are implied by the ancestors' start dimension values. So all the start dimension values of the node and its ancestors imply the dimension values in the node. In the range trie of our example(Figure 3), `(S1, P1)` $\rightarrow$ {D1} since the node `(P1, D1)` has the start dimension value `P1` and it has an ancestor `(S1, C1)` whose start dimension value is `S1`. Also, `(S1)` $\rightarrow$ {C1} because `S1` is the start dimension value of the node `(S1, C1)` and it has no ancestors.

Thus, the set of dimension values implied by `(S1, P1)` is {C1, D1}, since `(S1)` $\rightarrow$ {C1} leads to `(S1, P1)` $\rightarrow$ {C1} also. In general, given a node in a range trie, say, there exist k dimensions on the node and its ancestors, among which there are l start dimensions, we can get from Lemma 2 that the set of l start dimensions jointly implies those $k - l$ non-start dimensions.

Lemma 2 shows that a range trie identifies data correlation among different dimension values.Lemma 3 explains that data correlation captured in a range trie reveals cells sharing identical aggregation values.

Lemma 3 *Suppose we have an n-dimensional range trie, given any node, it has k dimensions existing on the node and its ancestors, that is, with dimension values $a_{i1}, a_{i2}..., a_{ik}$ on the dimensions $A_{i1}, A_{i2}..., A_{ik}$. Among these, l dimensions are the start dimensions of the node and its ancestors, say $A_{i1}, A_{i2}..., A_{il}$, where $l \leq k$. Let $c1$ be a $k-dimensional$ cell with dimension values $a_{i1}, a_{i2}, ..., a_{ik}$ on the dimensions $A_{i1}, A_{i2}..., A_{ik}$ and $c2$ be an $l-dimensional$ cell with dimension values $a_{i1}, a_{i2}..., a_{il}$ on the dimensions $A_{i1}, A_{i2}..., A_{il}$. Any c such that $c1 \preceq c \preceq c2$ will have the same aggregation value as $c1$ and $c2$.*

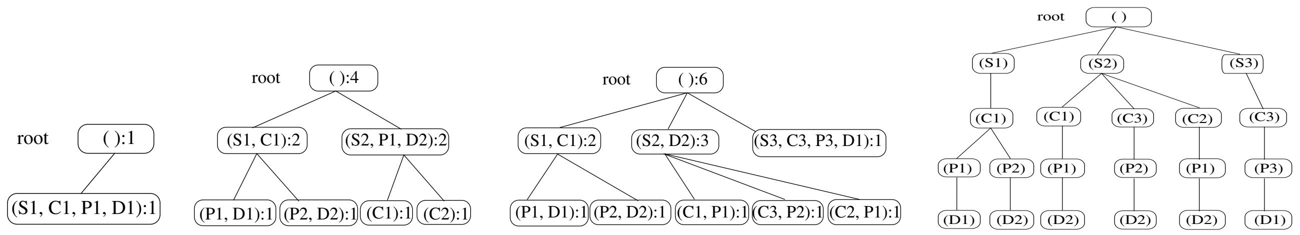
The intuition is straightforward. As we learned before, the set of start dimension values $(a_{i1}, a_{i2}...a_{il})$ imply those non-start dimension values $(a_{i(l+1)}, a_{i(l+2)}...a_{ik})$. The set of tuples used to compute the aggregation value of cell $c2$ are those having dimension values $(a_{i1}, a_{i2}...a_{il})$. According to Definition 5, all tuples with dimension values $(a_{i1}, a_{i2}...a_{il})$ have the dimension values $(a_{i(l+1)}, a_{i(l+2)}...a_{ik})$ on dimensions $A_{i(l+1)}, A_{i(l+2)}..., A_{ik}$. So, for any cell c, $c1 \preceq c \preceq c2$, the set of tuples to compute the aggregation value is the same as that to compute the aggragation value of $c2$.

In our example in Figure 3(c), all cells between `(S1, *, P1, *)` and `(S1, C1, P1, D1)` have identical aggregation values.

3.1 Constructing the Range Trie

This section will describe the range trie construction algorithm. It is done by a single scan over the input dataset like H-Tree. Each tuple will be propagated downward from the root till a leaf node. However, in an H-Tree or a star tree, each dimension value will be propogated on one node in order, while each node of a range trie extracts the common dimension values shared by the tuples inserted. For the set of dimension values not common to all tuples, choose the smallest dimension as the start dimension and insert the tuple to the branch rooted at the node whose start dimension value is equal to the tuple's dimension value.

Figure 4 gives the algorithm to construct a range trie from a given dataset. The algorithm iteratively inserts a tuple into the current range trie. Lines 3-26 describe how to insert dimension values of a tuple downward from the root of the current range trie. First, it will choose the way to go by searching for the child node whose start dimension value is identical with that of the tuple. If none of the existing child nodes has identical start dimension value as the tuple, the insertion is wrapped up by adding a new leaf



(a) The range trie after inserting `(S1,C1,P1,D1)`

(b) The range trie after inserting `(S1,C1,P2,D2)`, `(S2,C1,P1,D2)`, and `(S2,C2,P1,D2)`

(c) The range trie after inserting `(S2,C3,P2,D2)` and `(S2,C3,P2,D2)`

(d) The star tree or H-Tree

Figure 3. The construction of the range trie

node as a child of the root node, where all dimension values remaining in the tuple are used as the key of the new node (Line7,8). For example, when the last tuple `(S3, C3, P3, D1)` is inserted into the range trie shown in Figure 3(b), a new leaf node is created as the rightmost child node of the root in Figure 3(c).

Otherwise, it chooses the existing child node whose start dimension value is identical with that of the tuple. A set of common dimension values is obtained by comparing the dimension values of the tuple and those in the chosen child node, and is removed from the tuple. This is described in line 10-11 of the algorithm.

Now if the key of the chosen child node is all in the set of common dimension values, that is, all the dimension values in the chosen child node appear in the tuple, the root of a range trie is set to the chosen child node. The insertion continues in the next iteration to insert the tuple with fewer dimension values into the range trie rooted at the previously chosen node. If the tuple `(S1, C1, P3, D2)` is to be inserted in the range trie shown in Figure 3(b), the chosen child node `(S1, C1)` will not be changed. The dimension values left in the tuple will be `(P3, D2)` after removing `(S1, C1)`. The new iteration will insert the tuple `(P3, D2)` to the range trie rooted at the node `(S1, C1)`.

Lines 12-23 describes what to do if some dimension values in the chosen child node are not in the set of common dimension values. If the smallest dimension in the chosen node excluding the set of common dimensions is larger than the start dimension of the child nodes of the chosen node, append those non-common dimension values to all those child nodes(line 16). Then, the new iteration continues with the tuple with fewer dimensions and the range trie rooted at the chosen node. This is the case when the tuple `(S2, C3, P2, D2)` is inserted to the range trie in Figure 3(b). The child node `(S2, P1, D2)` is chosen to insert the tuple. The set of common dimension values is {`S2, D2`} on

Algorithm 1: Range Trie Construction

```
  Input:  a set of n-dimension data tuples;
  Output:  the root node of the new range trie;
  begin
1   create a root node whose key = $NULL$;
2   for each data tuple $a = (a_1, a_2, ..., a_n)${
3     currentNode = root;
4     while (there exist some dimension values in $a$) {
5       $a_i$=the value in $a$ on the start dimension of currentNode;
6       if no child of currentNode has start dimension value $a_i$
7         create a new node using dimension values in $a$ as key;
8         insert the new node as a child of currentNode; break;
9       else, {
10        commonKey=the set of values both in $c$ś key and $a$;
11        remove from $a$ the above common key values, commonKey;
12        if node $c$ has more key values than commonKey
13          diffKey = $c$->key - commonKey
14          set node $c'$s key value to commonKey);
15          if smallest dimension in diffKey>that of $c'$s children
16            append diffKey to $c'$s child nodes;
17          else
18            create a new node $c1$ using diffKey as key values;
19            set $c'$s children as $c1'$s children;
20            create another new node $c2$ using values left in $a$;
21            set $c1$ and $c2$as children of node $c$; break;
22          end if;
23        end if;
24        set currentNode to node $c$;
25      end if;
26    } end while
27  } end for
28  return root;
  end;
```

Figure 4. The range trie construction

the dimensions `Store, Date`. The smallest of the non-common dimensions is `Product` which is larger than the start dimension `City` of the child nodes of the chosen node, given the dimension order: `Store, City, Product, Date`. The set of non-common dimension values {`P1`} is appended to all the children, which become `C1, P1` and `C2, P1`. The tuple `(C3, P2)` is inserted in the range trie rooted at the updated chosen node `(S2, D2)` in the next iteration. This time no child node exists with the identical start dimension value as `C3`. The insertion ends after a new leaf node `(C3, P2)` is added as a child node of the current root node `(S2, D2)`, as shown in Figure 3(c). If the smallest of the non-common dimensions in the chosen node is smaller than the start dimension of its child nodes, create a new node which uses the set of non-common dimension values as the key and inherit all the child nodes of the chosen node as its children. Then, create another new leaf node which uses the remaining dimension values in the tuple as the key, and set these two new nodes as children of the chosen node. Here, if the chosen node is a leaf node itself, we assume the start dimension of its child nodes is alway largest. An example of this case occurs when the tuple `(S1, C1, P2, D2)` is inserted in the range trie shown in Figure 3(a). The non-common dimension values in the chosen node are `P1, D1` and thus non-common dimensions are `Product, Date`, where the smallest non-common dimension is `Product` according to the dimension order given before. It is smaller than the start dimension of the child nodes of the chosen node `(S1, C1, P1, D1)`, since it is a leaf node whose child nodes has the largest possible dimension as we assumed. The dimension values of the chosen node are replace by the common dimension values `(S1, C1)`, and a new node is created which has dimension values `(P1, D1)` and the child nodes of the chosen node is empty since the chosen node is a leaf node. Another new leaf node is created with the dimension values left in the tuple, `P2, D2`, and both new nodes are set as two child nodes of the updated chosen node. The result is shown as the left branch in the Figure 3(b).

The range trie constructed from a dataset is invariant to the order of data entry.The process of constructing a range trie is actually to iteratively find the common dimension values shared by a set of tuples, which is more than prefix sharing as in many other tree-like structure.

The Size of a Range Trie

As we will see later, the number of nodes in a range trie is an important indicator of the efficiency of the range cube computation. The range trie with fewer nodes will generally take less memory and less time to compute the cube, and the resulting cube will be more compressed. Suppose we have a range trie on a D-dimensional dataset with T tuples. So, it can have no more than T leaf nodes. The depth of the range trie is D in the worst case, when the dataset is very dense and the trie structure is like a full tree. It is $log_N T$ in the average case, for some N, the average fan-out of the nodes. Now we will give a bound on the number of internal nodes. The proof is by induction on the depth of the range trie.

Lemma 4 *The number of interior nodes in a range trie with T leaf nodes is bounded by $T-1$, while it is $\frac{T-1}{N-1}$ on average. Here, N is the average fan-out of the nodes.*

Compared to an H-tree, whose internal nodes are $T*(D-1)$ in the worst case, a range trie is more scalable w.r.t the number of dimensions,that is, its efficiency is less impacted by the number of dimensions.

4 Range Cube

As we have shown in Lemma 3, the range trie will identify a range of cells with identical aggregation values. The definition of a range coincides with the definition of sub-lattice.

Two cells c_1 and c_2 are needed to represent a range. However, since $c_1 \preceq c_2$, it means intuitively that either c_1 has the same dimension value as c_2 or c_1 has some value but c_2 has value "$*$" on each dimension. For each dimension value a, we introduce a new symbol $\overline{a}$ to represent either the value a or $*$. Thus, a range can be represented as a **range tuple** like a cell with n values followed by its aggregation value.

Definition 6 *Let $[a, b]$ be a range. If $a = (a_1, a_2, ..., a_n)$ is a $k-dimensional$ cell and $b = (b_1, b_2, ..., b_n)$ is an $l-dimensional$ cell, and $a \preceq b$. The* ***range tuple*** *c is defined as:*

$$c_i = \begin{cases} a_i & \textit{if } a_i = b_i \\ \overline{a_i} & \textit{if } a_i \neq * \textit{ and } b_i = * \end{cases}$$

A **range cube** is a partitioning of all cells in a cube and each partition is a range. All ranges in a range cube are disjoint and each cell must be in one range.

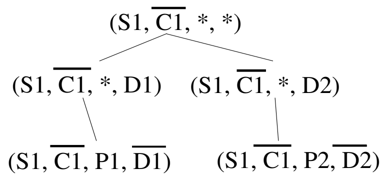

(S1, $\overline{C1}$, *, *)
(S1, $\overline{C1}$, *, D1)
(S1, $\overline{C1}$, *, D2)
(S1, $\overline{C1}$, P1, $\overline{D1}$)
(S1, $\overline{C1}$, P2, $\overline{D2}$)

Figure 5. The ranges with `Store` set to "`S1`"

A range cube is basically a compressed cube, which replaces a sequence of cells with a range. For example, `(S1, `$\overline{C1}$`, P1, `$\overline{D1}$`)` represents all the cells in the range `[(S1, *, P1, *), (S1, C1, P1, D1)]` which are: `(S1, *, P1, *)`, `(S1, C1, P1, *)`, `(S1, *, P1, D1)`, and `(S1, C1, P1, D1)`. So, the resulting cube size is reduced. As an example, the five ranges in Figure 5 consist of 14 cells.

A range cube has some desirable properties compared to other compressed cubes. First, it preserves the native reprentation of a data cube, i.e., each range is expressed as a tuple with the same number of dimensions as a cell. So, a wide range of current database and data mining applications can work with it easily. In addition, other compression or index techniques such as dwarf [14] can also be applied naturally to a range cube. This property makes it a good candidate to incorporate with other performance improvement approaches.

Second, as we will show, a range cube preserves the semantic properties of a data cube, which means it keeps the roll-up/drill-down property of a data cube as described in [10]. Lakshmanan et al. proposed the sematics-preserving property for a compressed cube in [10]. They developed several theorems to help prove the sematics-preserving property of a compressed cube. We will use them to show the sematics-preserving property of a range cube. First it is clear to see from the definition that a range cube is a convex partition of a full data cube. Now we reach the conclusion that a range cube is sematics-preserving.

Theorem 1 *A range cube preserves the roll-up/drill-down properties of a data cube.*

This follows from the formalization of partition preserving the roll-up/drill-down semantics of a cube in[10]. Lakshmanan et al.[10] show that (1)a partition is convex iff the equivalence relation inducing the partition is a weak congruence and (2)a weak congruence relation respects the partial order of cells in a cube. So it immediately leads to the conclusion that the range cube is a partition which respects the partial order of cells, that is, it preserves the roll-up/drill-down property of the original cube.

As an example, Figure 5 shows the roll-up and drill-down relation of the ranges on those cells whose `Store` dimension value is `''S1''`.

5 Range Cube Computation

The range cubing algorithm operates on the range trie and generates ranges to represent all cells completely. The dimensions of the data cube are ordered, yielding a list $A_1, A_2, ..., A_n$. It iteratively reduces a $n - dimensional$ range trie to a $(n-1) - dimensional$ range trie after traversing on the $n - dimensional$ range trie to generate ranges and recursively apply the range cubing on each node. The $n - dimensional$ range trie generates the cells in cuboids from $(A_1, *, ..., *)$ to $(A_1, A_2, ..., A_n)$. And then the $(n-1) - dimensional$ range trie generates cells in cuboids from $(*, A_2, ..., *)$ to $(*, A_2, ..., A_n)$ and so on.

5.1 An Example

In this section, we will illustrate the efficiency of range cube computation using the previous example shown in Fig 2(a). Fig 3(c) is the initial four-dimensional range trie constructed from the base table. The number inside each node is the number of tuples used to compute the aggregation values of each node. The algorithm visits the node `(S1, C1)` and produces the range (`S1`, $\overline{C1}$, $*$, $*$), representing the cells (`S1`, $*$, $*$, $*$) and (`S1, C1`, $*$, $*$). Then it applies range cubing algorithm on the subtrie rooted at at node `(S1, C1)`, which is defined on the dimension `Product, Date`. The range cubing on the sub-trie should generate the ranges containing all cells with dimension values either `(S1, *)` or `(S1, C1)`, that is, all the cells with dimension values `(S1)`. Similarly, the middle branch of the root will generate ranges representing all the cells with the dimension value `(S2)` and the right-most branch will generate those having the dimension value `(S3)`.

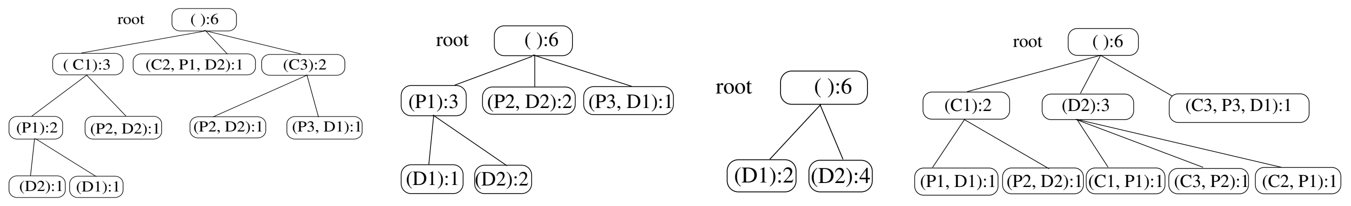
After traversing the initial range trie in Fig 3(c), it is reorganized to generate a three-dimensional range trie as shown in Fig 6(a). It is defined on the dimensions `City, Product, Date`. The way to reorganize a $n - dimensional$ range trie to a $(n-1) - dimensional$ range trie will be explained later. Similar traversal is applied to the three-dimensional range trie(Fig 6(a)), and it generates all the ranges including those cells with values `(*, C1) or (*, C2) or (*, C3)` on dimensions `Store` and `City`.

After that, the three-dimensional range trie is reduced to the two-dimensional range trie on dimensions `Product, Date` as shown in Fig 6(b). The transformation is obtained by reorganizing the three-dimensional range trie either. This time it produces all the ranges representing cells who have values `(*, *, P1)` or `(*, *, P2)` on dimensions `(Store, City, Product)`. Finally a one-dimensional range trie is obtained from the two-dimensional range trie as shown in Fig 6(c) and gives the aggregation values for the cells `(*, *, *, D1)` and `(*, *, *, D2)`.

Now we will explain how to transform a four-dimensional range trie to a three-dimensional range trie on `(City, Product, Date)`. By setting the `Store` dimension of each child nodes of the root to $*$, the trie will look like the one in Figure 6(d). Since the new start dimension is `City`, the child nodes which do not contain the dimension `Store` will append their dimension values to their children. That is, the dimension value `D2` is appended to its three children, which become `(C1, P1, D2)`, `(C3, P2, D2)`, `(C2, P1, D2)`. These three nodes will be merged with the node `(C1)` and the node `(C3, P3, D1)`. Then, the resultant three-dimensional range trie appears in Figure 6(a)

5.2 Range Cubing Algorithm

If we generalize the example given above, the algorithm can be obtained as in Figure 7. It starts from an initial $n - dimensional$ range trie, and does a depth-first traversal of the nodes. For each node, it will generate a range and recursively apply the cubing algorithm to it in case it is not a leaf node. Then it will reorganize the range trie to a $(n-1) - dimensional$ range trie and did the same thing on it. Similar things happen to the $(n-1) - dimensional$



(a) The range trie on dimensions `(City, Product, Date)`

(b) The range trie on dimensions `(Product, Date)`

(c) The range trie on dimensions `(Date)`

(d) A temporary trie in transition from $4 - dimension$ to $3 - dimension$

Figure 6. The process of range cube computation

range trie. After that, it comes to a $(n-2) - dimensional$ range trie, a $(n-3) - dimensional$ range trie and so on. The algorithm ends with a $1 - dimensional$ range trie.

Algorithm 2: Range Cube Computation
```
  Input:  root1:  the root node of the range trie;
        dim:  the set of dimensions of root1;
        upper:  the upper bound cell;
        lower:  the lower bound cell;
  Output:  the range tuples;
  begin
    for each dimension $iDim \in dim${
      for each child of root, namely, $c$,{
        Let startDim be the start Dim of the $c$;
        upper[startDim] = $c$ś start dimension value;
        set upper's dims $\in$ {$c$->key - startDim} to $*$;
        set lower's dims $\in$ $c$->key to $c$->KeyValue;
        outputRange(upper,lower, $c$->quant-info);
        if $c$ is not a leaf node
          call range cubing on $c$ recursively;
      }
      Traverse all the child nodes of the root and
      merge the nodes with same next startDim value;
      upper[iDim] = lower[iDim] = $*$;
      iDim = next StartDim;
    }
  end;
```

Figure 7. Range Cube Computation algorithm

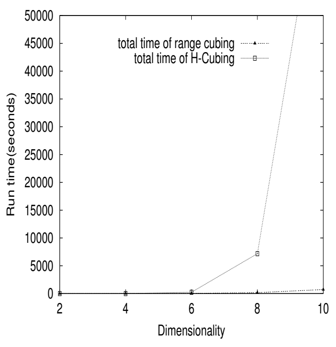
It is easy to see the number of recursive calls is equal to the number of interior nodes. So the bound on the interior nodes given previously provides a warranty of the computational complexity. Moreover, each node generated represents a distinct set of tuples to be aggregated for some cell(s), thus distinct aggregation values, so it can be shown that it needs least number of aggregation operations in the algorithm, once the initial range trie is constructed. The proof is not shown here due to the space constraint. The intuition is that each node represents a distinct aggragation values and each node representing some $n - dimensional$ cells is aggregated from those nodes representing $(n+1) - dimensional$ cells. In our example, the total number of aggregation needed is 9 in range cubing.

All existing cube computation algorithms are quite sensitive to dimension order since it determines the order to compute cells. The range trie also depends on the dimension order as shown before. However, it is only used to enforce the choice of start dimensions, that is, the start dimension of a child node must be larger than the start dimensions of its ancestors. The range trie provides some flexibility to dimension order since it not only allows the dimensions in a child nodes might be smaller than the non-start dimensions of the parents, but also allows different branch to have different dimension order based on the data distribution. This feature makes range cube computation less sensitive to dimension order.

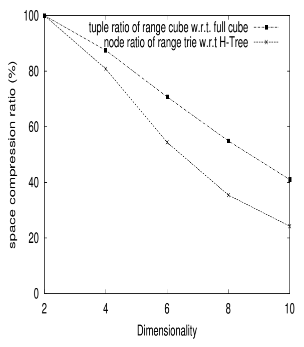
The favorite dimension order for the range cubing is also cardinality-descending, which is the same as star-cubing and BUC. Since the dimension with large cardinality is more possible to imply the dimension with small cardinality, it won't lead to much more number of nodes compared to the range trie based on the cardinality-ascending order. However, it produces smaller partition and thus achieves earlier pruning, while it also generates more compressed range cube.

6 Experimental results

To evaluate the efficiency of the range cubing algorithm in terms of time and space, we conducted a comprehensive set of experiments and present the results in this section. Currently there exist two main cube computation algorithms: BUC[4] and H-Cubing[9],that is shown to outperform BUC in cube computation. We cannot do a thorough comparison with the latest star cubing algorithm, which is to appear in VLDB03, due to time limit. We would like to include it in the near future. In this paper, we will only report result comparing with H-Cubing and more results with comparison with star-cubing and condensed cube will be included soon.As we will see, the experiments show that range cubing saves computation time substantially, and reduces the cube size simultaneously. The range cube does

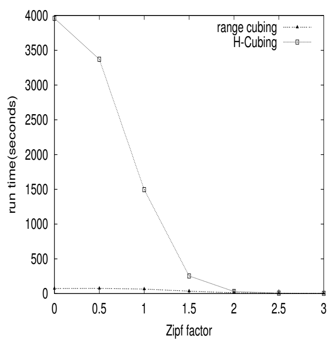


(a) comparison of total run time

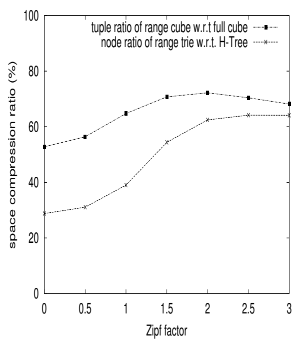


(b) space compression of range cube

Figure 8. Evaluating the effectiveness of range cubing



(a) comparison of total run time



(b) space compression of range cube

Figure 9. Evaluating the impact of skewness

not try to compress the cube optimally like Quotien-Cube which finds all cells with identical aggregation values, however, it still compresses the cube close to optimality as shown in the case of real dataset. It balances the computational efficiency and space saving to achieve overall improvements.

Both synthetic and real dataset are used in the experiments. We used uniform and Zipf[18] distributions for generating the synthetic data. These are standard datasets most often used to test the performance of cube algorithms [10, 15, 4]. The real dataset is the data of weather conditions at various weather land stations for September 1985 [8]. It has been frequently used in gauging the performance of cube algorithms [10, 15, 12]. We ran the experiments on an AthlonXP 1800+ 1533MHz PC with 1.0G RAM and 50G byte hard disk.

We measure the effectiveness of range cubing on the basis of three metrics. The first one is the total run time to generate the range cube from an input dataset, which measures the computational cost. The results about computation time without I/O are omitted since its improvement is no worse than the total run time. The second one is tuple ratio[15], which is the ratio between the number of tuples in the range cube and the number of tuples in the full cube. Tuple ratio measures the space compression of the resulting cube, and the smaller the ratio the better. The third metric is the *node ratio*, which is defined as the ratio between the number of nodes in the initial range trie and that in the H-Tree. The number of nodes is an important indicator of the memory requirement of the cube computation. For a specific dimension order, the less the number of nodes, the better the performance.

6.1 Synthetic Dataset

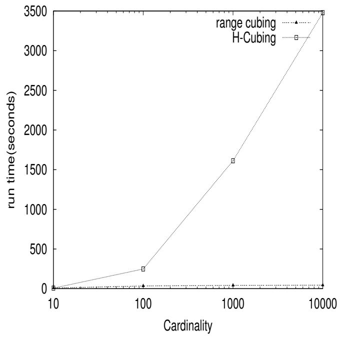
First, we demonstrate the effectiveness of range cubing and then show the performance in case of skewness and sparsity. Finally we will investigate its scalability. ndfigure

Effectiveness. This set of experiments shows the effectiveness of range cubing in reducing time and space costs. We use a synthetic dataset with Zipf distribution. The Zipf factor is fixed at 1.5. We also fix the number of tuples at 200K and the cardinality of each dimension is 100. The number of dimensions is varied from 2 to 10. Fig 8(a) shows the timing result and Fig 8(b) shows the space compression result.

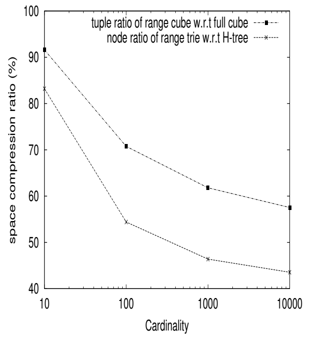
When the dimensionality increases, both range cubing and H-Cubing need more time and space. However, the possibility of data correlation increases too. So Range Cubing does not grow as rapidly as H-Cubing does, and achieves better time and space compression ratios. The computation time of range cubing is one eighth of H-Cubing even with a 6-dimension dataset. The space ratio improves as the dimensionality grows.

When the cube is very dense (2 to 4 dimensions), the computation time and cube size of range cubing and H-Cubing are almost the same. This shows that the lower bound of a range trie is an H-Tree and a range cube in the worst case is an uncompressed full cube.

Skewness. We investigate the impact of the data skew in the second set of experiments. We set the number of dimensions to 6 and the cardinality of each dimension to 100 and the number of tuples to 200K. We generate the dataset with the Zipf factor ranging from 0.0(uniform) to 3.0(highly skewed) with 0.5 interval. Fig 9(a) and Fig 9(b) show how the computation time and the cube size compression change with regard to increased data skew.



(a) comparison of total run time



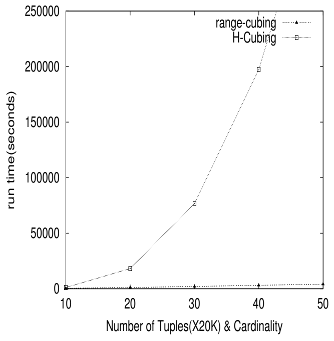
(b) space compression of range cube

Figure 10. Evaluating the impact of sparsity

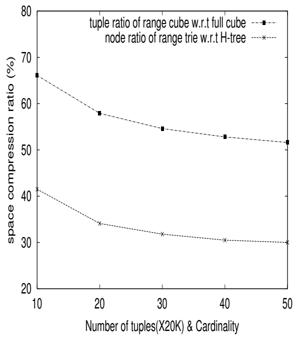
As data gets more skewed, both cubing algorithms decrease in their computation time, because their data structures adapt to the data distribution. The space compression ratio increases with Zipf factor and it becomes stable after Zipf factor reaches 1.5. As data gets skewed, the space compression in a smaller dense region decreases while the space compression in the sparse region increases. When the Zipf factor is about 1.5, the two effects balance. As noted for BUC, and BUC-dup algorithms[4], they perform worse as Zipf factor increases, and perform worst when Zipf factor is 1.5.

Sparsity. To evaluate the effect of sparse data, we vary the cardinality of dimensions while fixing other parameters. This is different from most prior experimental evaluations[10, 15], which vary the number of tuples while fixing the cardinality. The reason is that the latter approach changes both the sparsity of the dataset and the experimental scale (the size of dataset). When we measure the cube space compression ratio, we can assume that when we enlarge a dataset with a specific level of sparsity, the space compression ratio will remain stable. However, a similar assumption cannot be made for the run time. Hence, in order to isolate only the sparsity effect, we vary the cardinality instead of the number of tuples. The results in Fig 10 are based on the dataset with Zipf factor of 1.5, the number of dimensions is 6, and the number of tuples is 200K. The cardinality of the dimensions takes the values 10, 100, 1000 and 10000.

When the cardinality gets larger, the run time used by H-Cubing algorithm increases rapidly while that of range cubing does not change much. The space compression ratio improves. The reason is that more data *coincidence* happens when the data is sparse, resulting in a more compressed range trie. This means that on average, a range tuple represents more cells. Meanwhile, the efficiency of h-cubing comes from prefix sharing in the h-tree structure. It has less prefix sharing as the cardinality increases, which makes its preformance worse with larger cardinality.



(a) comparison of total run time



(b) space compression of range cube

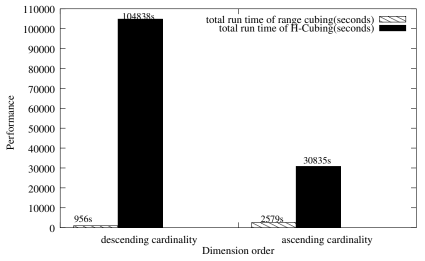
Figure 11. Evaluating the scalability of algorithms

Scalability. We also measured how the algorithm scales up as the dataset size grows. We vary both the number of tuples and the cardinality, in order to accurately measure the effect of experiment scale on the cubing algorithm. Most prior research changed only the number of tuples to measure scalability. However, this will cause the dataset to get denser simultaneously. Then the impact on the algorithm is a hybrid of the increased density and the experiment size. Therefore, we change both so that the dataset density remains stable as the dataset size increases. In our experiments, we use a dataset with 10 dimensions and Zipf factor 1.5. The cardinality ranges from 10 to 50 with 10 interval. The number of tuples varies from 200K to 1M with the interval of 200K.

The results in Fig 11 shows that range cubing is much more scalable. The total run time in case of 200K tuples with cardinality 10 for each of 10 dimensions is 412.3 seconds for range-cubing and 1072.5 for H-Cubing. While H-cubing goes up rapidly and reaches more than 387803 seconds in case of 1M tuples with cardinality 50; Range cubing in this case takes only 4104 seconds. As we can see, the space compression ration is slightly better when the experiment scale goes up, since the data density does not change.

6.2 Real Dataset

We used the weather dataset which contains 1,015,367 tuples. The attributes with cardinalities are as follows: station-id (7,037), longitude (352), solar-altitude (179), latitude (152), present-weather (101), day (30), weather-change-code (10), hour (8) and brightness (2).



Dimension Order	Tuple Ratio (#ranges / #cells)
descending cardinality	11.2%
ascending cardinality	64%

Figure 12. Evaluating performance on the real dataset

The dimension order of the dataset does affect the performance in the range cubing algorithm. As mentioned before, it prefers to order dimensions by decreasing cardinality. The reason is that it allows flexible dimension orders and may have different dimension order on each branch. sensitive to dimension order than other algorithms.

As we can see from the results in Fig 12, the range cubing algorithm takes significantly less time and space than H-Cubing. When we order the dimensions by the decreasing cardinality,as favored by range cubing, it is more than 30 times faster than H-Cubing in its favorable dimension order (by increasing cardinality) and generates a range cube with only 11.2% of the full cube size. Even in the least favorable dimension order (increasing cardinality), range cubing takes less than one fortieth of the run time used by H-Cubing in its least favorable order. The space compression ratio of the range cube grows to 64% in its least favorable dimension order.

7 Discussion and Future Work

This paper proposes a new cube computation algorithm, which utilizes the correlation in the datasets to effectively reduce the computational cost. Although it does not compress the cube optimally, its goal is to achieve overall speed-up by balancing computational efficiency and space efficiency. It not only substantially reduces computational time of the cube construction, but also generates a compressed form of the cube, called the range cube. The range cube is a partition of the full cube and effectively preserves its roll-up/drill-down semantics. Moreover, a range is expressed as a tuple similar to a cell, so that it preserves the format of the original dataset. These desirable properties facilitates integration of range cube with existing database applications and data mining techniques.

Since the range trie reduce the effective dimensions of input datasets, it should good at handling high-dimensional sparse or correlated dataset. Even when the dataset is fully dense, the range trie degrades to the H-Tree, and the range cube is a full cube.

Our current efforts include incorporating constraints with range cube computation, dealing with holistic functions, and applying some other compression techniques to compress the cube. Supporting incremental and batch updates, are to be explored in the future.

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